A statistical comparison of the Vienna Game Opening using rate-of-change, integral, and static evaluations.

Exam Session: May 2021

Course: IB Math AA SL

Word Count: 2426

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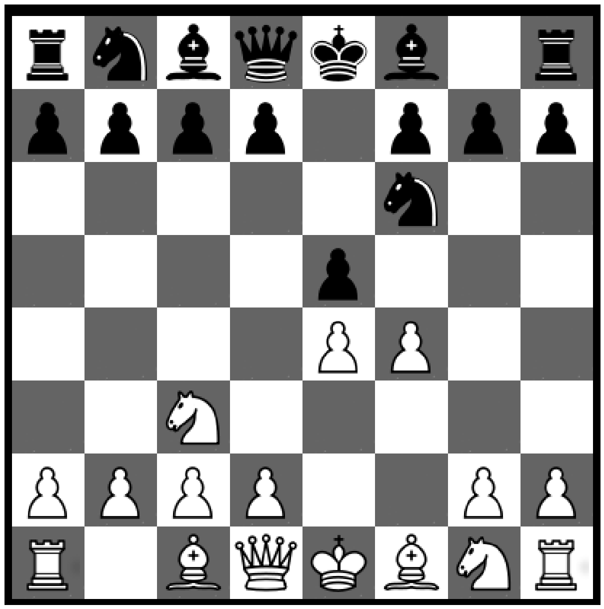


Vienna Gambit

In the last three months, I've gotten back into a hobby I enjoyed in my middle school years: chess. My favorite opening, referring to a common set of moves played in the start of the match, is the Vienna Game (pictured above). When I was in middle school, I was able to memorize different games where the opening was played; its intricacies and motives. Now, however, this is not the case. Armed with a better understanding of mathematical concepts like differentiation and integration and a better mathematical vocabulary, I am interested in evaluating both openings and their many variations using integration and differentiation techniques to better gain an intuitive understanding of which opening line is better and the use-cases for integrating and differentiating.

Chess Basics

Because this paper involves a study of chess, a brief introduction to some of the vocabulary is found below.

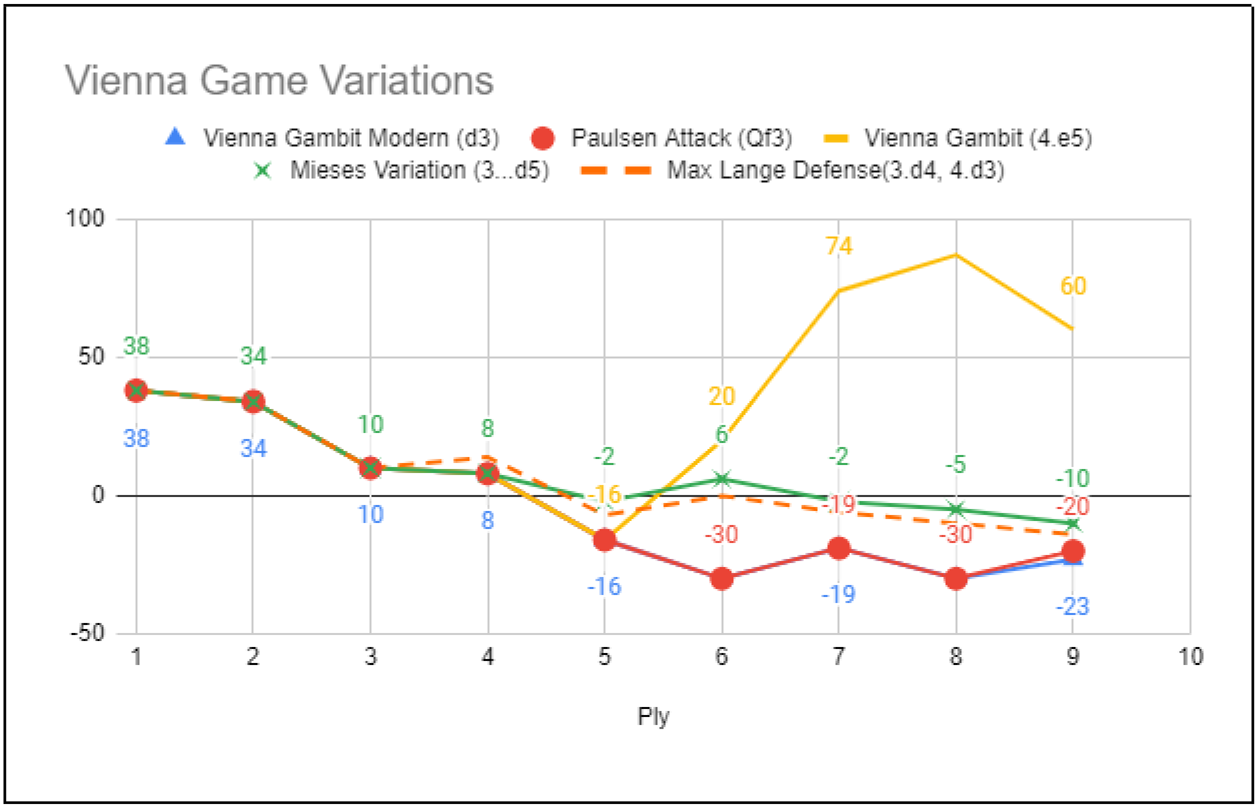
Ply refers to a half-turn. From the mathematical perspective, a *turn* is like the period of oscillation for a simple harmonic oscillator; when the turn comes back to the first player is said that one turn has been completed.

Algebraic Notation refers to notation used for discussing chess movements. An example of algebraic notation is, “e4”. “e4” translates to the pawn on e2 being moved to e4. Generally speaking, the last two characters represent the target destination, the first character represents the abbreviation for the piece, the inclusion of ‘x’ indicates a capture, and ‘O-O’/‘O-O-O’ both represent a casted king. Below are more examples of algebraic notation:

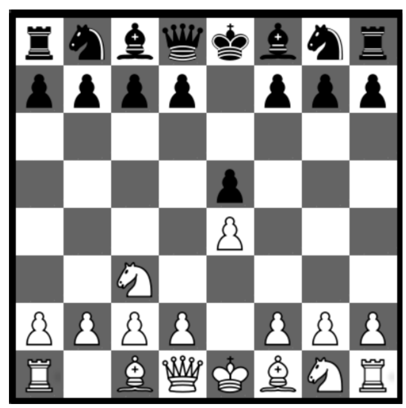
Ne2 - “Knight to e2”
Bc4 - “Bishop to c4”
Qxh4 - “Queen takes the piece and moves to h4”
O-O - “King castles short (away from the direction of Queen’s original square)”
O-O-O - “King castles long (in direction of Queen’s original square)”

Evaluation refers to a numerical representation of the “goodness” of a position. I am using a very powerful chess engine called *StockFish* to evaluate the games. The evaluation is evaluated through agreed heuristics such as a king’s position, the number of pieces for each player, the stage of the game, etc.

The Vienna Game



The Vienna Game is characterized by the moves e4[1], e5[2], and Nc3[3], as pictured to the right. When playing the Vienna Game, there are many interesting variations where both white and black can gain quick advantages in the start of the game. The knight on c3 is protecting the e4 pawn and black's options are to create an attack on the e4 pawn, or to play more passively and defend their e5 pawn.



Falkbeer Variation

The Falkbeer Variation of the Vienna Game occurs when black responds by playing Nf6[4]. A common continuation of the Falkbeer Variation is known as the Vienna Gambit where white plays f4[5]. In the graph pictured above, the evaluation of the Vienna Gambit (Modern) can be represented by the following piecewise formula:

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[1] e4, can also be interpreted as (5, 4) in coordinate notation
[2] e5, can also be interpreted as (5, 5) in coordinate notation
[3] Nc3, refers to the knight on (3,3)
[4] Nf6, knight (6, 6)
[5] f4, pawn to (6, 4)

$$f(x) = \begin{cases} -4.0x + 42.0 & 1 \le x \le 2 \\ -24.0x + 82.0 & 2 \le x \le 3 \\ -2.0x + 16.0 & 3 \le x \le 4 \\ -24.0x + 104.0 & 4 \le x \le 5 \\ -14.0x + 54.0 & 5 \le x \le 6 \\ 11.0x + -96.0 & 6 \le x \le 7 \\ -11.0x + 58.0 & 7 \le x \le 8 \\ 7.0x + -86.0 & 8 \le x \le 9 \end{cases}$$

Vienna Gambit: Modern

Ply	Vienna Gambit Modern (d3)
1	38
2	34
3	10
4	8
5	-16
6	-30
7	-19
8	-30

The right-most column in the table represents the evaluation after having made the best move. The notation for the moves was not included in this table, and is rather in the appendix. The piecewise formula was derived as such:

Deriving line for ply 1 - 2

$$y - y_1 = m(x - x_1)$$

$$y - y_1 = (\frac{y_2 - y_1}{x_2 - x_1})(x - x_1)$$

$$y - 34 = (\frac{34 - 38}{2 - 1})(x - 2)$$

$$y - 34 = (\frac{34 - 38}{2 - 1})(x - 2)$$

$$y - 34 =- 4(x - 2)$$

$$y - 34 =- 4x + 8$$

$$y =- 4x + 8 + 34$$

$$y =- 4x + 42$$

The derivation to the left uses the formula for a line that crosses two points to derive the "line of best fit" across points 1-2. As can be noted in the graph with the Vienna Variations, Vienna Gambit Modern (d5) does follow the trend outlined by its piecewise function above. The function was derived in this way as a result of the discrete values. The same was done for each variation of the Vienna Opening, and the functions are included in the appendix.

The table below summarizes the information found in the Vienna Variations graph:

	Average Eval	Final Eval
Vienna Gambit Modern (d3)	-3.11	-23.00
Paulsen Attack (Qf3)	-2.78	-20.00
Vienna Gambit (4.e5)	35.00	60.00
Mieses Variation (3...d5)	8.56	-10.00
Mieses Variation (3...Bc5)	12.11	-4.00
Max Lange Defense(3.d4, 4.d3)	6.56	-14.00

As can be seen, the Vienna Gambit (4.e5) is the strongest line in the variation where the player playing as white pushes their pawn to the e5 square threatening the knight on c6. Unfortunately, this variation is rather well known and is usually not played often. Thus the Mieses Variation, with the Bc5 move instead of the f4 pawn push is a better variation for white with a final evaluation of -4.00 and average evaluation of 12.11; second to the Vienna Gambit.

From a statistical point of view, however, the evaluation does not hint as to how the specific variations might continue in moves to come. Take for example the Vienna Gambit (4. e5), it appears to be the worst until ply 6 when the gambit is accepted (the pawn taken). To better understand the openings, I am interested in evaluating the openings by differentiating and integrating the piecewise functions.

Differentiating the Vienna Game: Mieses Variation (3...Bc5)

From a practical perspective, it is better to focus on the Mieses Variation since it is a sound opening and more common; appearing 13% more in my game database than the Vienna Gambit (4. e5).

$$f(x) = \begin{cases} -4.0x + 42.0 & 1 \leq x \leq 2 \\ -24.0x + 82.0 & 2 \leq x \leq 3 \\ -2.0x + 16.0 & 3 \leq x \leq 4 \\ -10.0x + 48.0 & 4 \leq x \leq 5 \\ 14.0x + -72.0 & 5 \leq x \leq 6 \\ -11.0x + 78.0 & 6 \leq x \leq 7 \\ 11.0x + -76.0 & 7 \leq x \leq 8 \\ -16.0x + 140.0 & 8 \leq x \leq 9 \end{cases}$$ Vienna Game: Mieses Variation (3… Bc5) [Non-differentiated]	(table below)

Ply	Mieses Variation (3...Bc
1	38
2	34
3	10
4	8
5	-2
6	12
7	1
8	12
9	-4

Differentiating

Because it is a piecewise function, the first order differential of the function $f(x)$, is equivalent to the first order differential of all of the inner functions such that $f'(x)$is equivalent to the function below:

$$f(x) = \begin{cases} -4.0 & 1 \leq x \leq 2 \\ -24.0 & 2 \leq x \leq 3 \\ -2.0 & 3 \leq x \leq 4 \\ -10.0 & 4 \leq x \leq 5 \\ 14.0 & 5 \leq x \leq 6 \\ -11.0 & 6 \leq x \leq 7 \\ 11.0 & 7 \leq x \leq 8 \\ -16.0 & 8 \leq x \leq 9 \end{cases}$$

An example of the work done when differentiating the above functions is shown below:

$$f'(x_{1-2}) = -4.0x + 42.0$$

$$f'(x_{1-2}) = -(1)4.0x^{1-1}$$

$$f'(x_{1-2}) = -(1)4.0x^{0}$$

$$f'(x_{1-2}) = -(1)4.0(1)$$

$$f'(x_{1-2}) = -4.0$$

The formula above represents the derivative of the function $f(x)$, where $f(x)$ represents the evaluation for turn, x. Because the derivative describes the slope, it is intuitive that the differentiated equation yields the change in evaluation at ply x. Using this technique, the table below demonstrates the average rate of change of the evaluations for each opening as well as the rate of change, at the final ply.

	Average de/dx	Final de/dx
Vienna Gambit Modern (d3)	-6.78	7.00
Paulsen Attack (Qf3)	-6.44	10.00
Vienna Gambit (4.e5)	2.44	-27.00
Mieses Variation (3...d5)	-5.33	-5.00
Mieses Variation (3...Bc5)	-4.67	-16.00
Max Lange Defense(3.d4, 4.d3)	-5.78	-4.00

As can be noted from the de/dx, both average and final, the Vienna Gambit (4.e5) appears to provide the highest average rate of change. As a matter of fact, it is the only variation that appears to favor white. The final de/dx, however, is the worst for the Vienna Gambit (4. e5).

In this case, the de/dx as a mathematical tool for analysis would indicate that the evaluation is either changing in the positive (in favor of white) or negative (in favor of black), but the issue here is that it does not take into account the moves available on the next turn. On the other hand, the average de/dx hints at the openings' favorability for a certain side, with a positive average de/dx signifying a net positive change in evaluation, and vice-versa.

Regarding the results, the normal evaluation provided ample proof that the Mieses Variation, with it being the second best variation, was a sold variation to use. When looking at the de/dx data, however, the Mieses Variation (3… Bc5) has the second lowest final de/dx! As a matter of fact, using the de/dx data, there is ample evidence that gambits, when you sacrifice a piece in return for development, tend to lead to higher average de/dx as was the case for the Vienna Gambit (4. e5) and higher final de/dx as is the case with the Vienna Gambit Modern (d3). Interestingly, Maxime Vachier-Lagrave vs Gilles Lietard in 2000 featured a game where Maxime, a current French grandmaster, played the Vienna Gambit: Modern and decisively won the game.

Integrating

The integration of the piecewise function can be done by taking the antiderivative and using power rules, etc. An example of a Rhiemman sum is shown below for calculating the integral of the Mieses Variation (3...Bc5):

$$f(x) = \begin{cases} -4.0x + 42.0 & 1 \leq x \leq 2 \\ -24.0x + 82.0 & 2 \leq x \leq 3 \\ -2.0x + 16.0 & 3 \leq x \leq 4 \\ -10.0x + 48.0 & 4 \leq x \leq 5 \\ 14.0x + -72.0 & 5 \leq x \leq 6 \\ -11.0x + 78.0 & 6 \leq x \leq 7 \\ 11.0x + -76.0 & 7 \leq x \leq 8 \\ -16.0x + 140.0 & 8 \leq x \leq 9 \end{cases}$$

Mieses Variation (3… Bc5)

The work for calculating the area under the curves is shown below:

$$\int_1^2 f(x) =- 4.0x^2(\frac{1}{2}) + 42.0x + C$$

$$\int_1^2 f(x) = (- 4.0(2)^2(\frac{1}{2}) + 42.0(2) + C) - (- 4.0(1)^2(\frac{1}{2}) + 42.0(1) + C)$$

$$\int_1^2 f(x) = (- 8.0 + 84.0 + C) - (- 2.0 + 42.0 + C)$$

$$\int_1^2 f(x) = (76 + C) - (40.0 + C)$$

$$\int_1^2 f(x) = 76 - 40.0$$

$$\int_1^2 f(x) = 76 - 40.0$$

$$\int_1^2 f(x) = 36$$

The above calculation demonstrates the integration from 1 to 2 of the Mieses Variation. The same steps were performed to all of the sub-functions and variations to get to the table below:

	Net Evaluation
Vienna Gambit Modern (d3)	-35.50
Paulsen Attack (Qf3)	-14.00
Vienna Gambit (4.e5)	266.00
Mieses Variation (3...d5)	63.00
Mieses Variation (3...Bc5)	92.00
Max Lange Defense(3.d4, 4.d3)	47.00

From the integration table, it is evident that the best variation is the Vienna Gambit (4.e5) followed by the Mieses Variation (3...Bc5) as the other evaluation methods also predicted. Unlike the de/dx or standard evaluation, the integral evaluation lends a much clearer picture regarding the long term success of each variation via the total evaluation each line yields.

The integral evaluation however, is also flawed as it doesn't take into account anything about the consistency of the evaluation through the turns. For instance, the Paulsen Attack (Qf3), which is ranked the second worst, can lead to serious complications for black if white is later allowed to castle and get the rook and queen on the same file starting at the f7 square (which is a very weak square for the black king).

Conclusion

	Average Eval	Final Eval	Average de/dx	Final de/dx	Net Evaluation
Vienna Gambit Modern (d3)	-3.11	-23.00	-6.78	7.00	-35.50
Paulsen Attack (Qf3)	-2.78	-20.00	-6.44	10.00	-14.00
Vienna Gambit (4.e5)	35.00	60.00	2.44	-27.00	266.00
Mieses Variation (3...d5)	8.56	-10.00	-5.33	-5.00	63.00
Mieses Variation (3...Bc5)	12.11	-4.00	-4.67	-16.00	92.00
Max Lange Defense(3.d4, 4.d3)	11.56	-14.00	-5.78	-4.00	47.00

The table above demonstrates the overall findings, and the results lead me to believe that the **Vienna Gambit (4. e5)** is the best variation of the ones listed. It should be noted that the Vienna Gambit (4.e5) is also known as the **Vienna Gambit: Accepted**, where 4. e5 is played. It is important to note, however, that the Vienna Gambit is only useful when accepted. With correct play from the opponent's side, the gambit simply results in an awkward and dubious position for white that can quickly fall apart since the f4 push that is so critical to the Vienna Gambit also allows for moves like Qh4+, that if not guarded against correctly, can lead to dangerous play. Below is a summary of all the variations and their statistical analysis.

Vienna Gambit Modern

The Vienna Gambit Modern is a close relative of the Vienna Gambit (4. e5), yet it ranks as the worst in average evaluation, final evaluation, average de/dx, and net evaluation. The Vienna Gambit Modern goes as such: *e4*, *e5*, *Nc3*, *Nf6*, *f4*, *d5*, *fxe5*, *Nxe4*, *d3*, etc. The Vienna Gambit Modern, however, places second in the final de/dx evaluation; which would imply that there is more at stake for white in this position than the classical Vienna Gambit with 4.e5 as opposed to 4. fxe5. Indeed, the game played by Maxime Vachier-Lagrave is a clear example of how such games can go[6].

Paulsen Attack (Qf3)

The Paulsen Attack occurs after the moves *e4*, *e5*, *Nc3*, *Nf6*, *f4*, *d5*, *fxe5*, *Nxe4*, *Qf3*. Both the Vienna Gambit Modern and the Paulsen Attack share the first 8 moves with the differentiating move being the response to the *Nxe4*. From the evaluations listed in the table above, it is clear that the engine prefers the Paulsen Attack over the Modern Defense, since it takes advantage of the queen's mobility.

Vienna Gambit (4. e5)

This is a classical variation of the Vienna Gambit and it goes *e4*, *e5*, *Nc3*, *Nf6*, *f4*, *exf4*, *e5*, *Ng8*, *Nf3*. From a statistical point of view, the Vienna Gambit, when it is accepted with *exf4* performs really well due to black's hindered mobility and the pawn's threat on the *f6* knight. It is clear from the stats, that the Vienna Gambit's

[6] Maxime Vachier-Lagrave vs Gilles Lietard (2000) (chessgames.com)

advantage lies with it being accepted as it performs the best in all categories except for the final de/dx. This is in part due to the fact that it is not clear how white can advance. There are ideas like Bc4, and then O-O, but black is able to refute some of these threats with their own set of moves.

Mieses Variation (3...d5)

The Mieses Variation goes *e4*, *e5*, *Nc3*, *Nf6*, *g3*, *d5*, *exd5*, *Nxd5*, *Bg2*. This variation of the Vienna Game is characterized by the *g3* pawn push. The *g3* pawn push is done with a series of ideas such as positioning the bishop on g2 and transposing into the King's Indian, or other such openings. From a statistical point of view, this line of the Mieses Variation where black immediately attempts to threaten the *e4* pawn is mediocre for white.

Mieses Variation (3… Bc5)

The Mieses Variation goes *e4*, *e5*, *Nc3*, *Nf6*, *g3*, *Bc5*, *Bg2*, *Nc6*, *d3*. This variation of the Mieses Variation is a semi-closed system where white and black are both increasing the tension on the board by developing their pieces to key squares, but the threats on both sides are nullified. This is evident in the evaluation, since it is a better alternative to the 3… d5 variation above by a factor of 2.2 (final evaluation).

Max Lange Defense (3.d4, 4.d3)

All of the Vienna systems described above are based on the Falkbeer variation where black responds to white's Nc3 with Nf6. In the Max Lange Defense, black instead opts to respond with Nc6; playing a mirrored game.In average evaluations the Max Lange Defense is similar to the Mieses Variation 3… Bc5, but it does not hold as well in the final evaluation, with the de/dx results indicating that it gets worse.

Evaluating these opening variations using derivative and integral techniques provide an interesting view as to the efficacy of specific lines. For example, the Vienna Gambit (4. e5) is the best when accepted, but the chances of it being accepted decrease as you face more experienced players. Therefore, a less riskier variation could be the Mieses Variation, with it’s mediocre overall performance and solid *g3* lines that allow for the *Bg2* later on. As a player, my understanding of this opening has evolved, and see that the use-cases for differentiating and integrating are both different and yet useful in their own way.

Appendix

$$f(x) = \begin{cases} -4.0x + 42.0 & 1 \leq x \leq 2 \\ -24.0x + 82.0 & 2 \leq x \leq 3 \\ -2.0x + 16.0 & 3 \leq x \leq 4 \\ -24.0x + 104.0 & 4 \leq x \leq 5 \\ -14.0x + 54.0 & 5 \leq x \leq 6 \\ 11.0x + -96.0 & 6 \leq x \leq 7 \\ -11.0x + 58.0 & 7 \leq x \leq 8 \\ 10.0x + -110.0 & 8 \leq x \leq 9 \end{cases}$$

Vienna Game: Paulsen Attack

$$f(x) = \begin{cases} -4.0x + 42.0 & 1 \leq x \leq 2 \\ -24.0x + 82.0 & 2 \leq x \leq 3 \\ -2.0x + 16.0 & 3 \leq x \leq 4 \\ -10.0x + 48.0 & 4 \leq x \leq 5 \\ 8.0x + -42.0 & 5 \leq x \leq 6 \\ -8.0x + 54.0 & 6 \leq x \leq 7 \\ -3.0x + 19.0 & 7 \leq x \leq 8 \\ -5.0x + 35.0 & 8 \leq x \leq 9 \end{cases}$$

Mieses Variation (3....d5)

$$f(x) = \begin{cases} -4.0x + 42.0 & 1 \leq x \leq 2 \\ -24.0x + 82.0 & 2 \leq x \leq 3 \\ 4.0x + -2.0 & 3 \leq x \leq 4 \\ -21.0x + 98.0 & 4 \leq x \leq 5 \\ 7.0x + -42.0 & 5 \leq x \leq 6 \\ -6.0x + 36.0 & 6 \leq x \leq 7 \\ -4.0x + 22.0 & 7 \leq x \leq 8 \\ -4.0x + 22.0 & 8 \leq x \leq 9 \end{cases}$$

Max Lange Defense

$$f(x) = \begin{cases} -4.0x + 42.0 & 1 \leq x \leq 2 \\ -24.0x + 82.0 & 2 \leq x \leq 3 \\ -2.0x + 16.0 & 3 \leq x \leq 4 \\ -24.0x + 104.0 & 4 \leq x \leq 5 \\ 36.0x + -196.0 & 5 \leq x \leq 6 \\ 54.0x + -304.0 & 6 \leq x \leq 7 \\ 13.0x + -17.0 & 7 \leq x \leq 8 \\ -27.0x + 303.0 & 8 \leq x \leq 9 \end{cases}$$

Vienna Gambit (4.e5)

	Evaluation					
Ply	Vienna Gambit Modern (d3)	Paulsen Attack (Qf3)	Vienna Gambit (4.e5)	Mieses Variation (3...d5)	Mieses Variation (3...Bc5)	Max Lange Defense(3.d4, 4.d3)
1	38	38	38	38	38	38
2	34	34	34	34	34	34
3	10	10	10	10	10	10
4	8	8	8	8	8	14
5	-16	-16	-16	-2	-2	-7
6	-30	-30	20	6	12	0
7	-19	-19	74	-2	1	-6
8	-30	-30	87	-5	12	-10
9	-23	-20	60	-10	-4	-14